LESSON DETAILS

Finding Infinity

Lesson Summary

This lesson introduces students to relations of the form $x + y = k$, $2x + 2y = k$ and $xy = k$ and uses these graphs as a platform to explain infinity and density. Throughout the lesson students will create graphs, both by hand and using technology, and will strengthen their understanding of how to use lines and/or curves of best fit to uncover information about the relation.

Grade: 9

Big Ideas

Graphs of Relations, Infinity and Density

Learning Expectations

AA1. develop and explore a variety of social-emotional learning skills in a context that supports and reflects this learning in connection with the expectations across all other strands

- building healthy relationships and communicating effectively in mathematics
- developing critical and creative mathematical thinking

A1. apply the mathematical processes to develop a conceptual understanding of, and procedural fluency with, the mathematics they are learning

- representing
- connecting

A2. make connections between mathematics and various knowledge systems, their lived experiences, and various real-life applications of mathematics, including careers

B1. demonstrate an understanding of the development and use of numbers, and make connections between sets of numbers

B1.3 use patterns and number relationships to explain density, infinity, and limit as they relate to number sets

C4. demonstrate an understanding of the characteristics of various representations of linear and non-linear relations, using tools, including coding when appropriate

C4.2 graph relations represented as algebraic equations of the forms $x = k$, $y = k$, $x + y = k$, $x - y = k$, $ax + by = k$, and $xy = k$, and their associated inequalities, where a, b, and k are constants, to identify various characteristics and the points and/or regions defined by these equations and inequalities

D1. describe the collection and use of data, and represent and analyse data involving one and two variables

D1.3 create a scatter plot to represent the relationship between two variables, determine the correlation between these variables by testing different regression models using technology, and use a model to make predictions when appropriate

Cross Curricular Connections

Learning Goals and Success Criteria:

The following are suggested Learning Goals and Success Criteria. Ideally, the teacher will co-construct these with their students. By inviting students to help define what success looks like in the lesson, student voice is included and students know what they need to do. These goals should be shared after the Minds-On section of the lesson.

LG1: We are learning to use patterns to explore the shape of graphs.

SC1: I can give an example of a pattern that will produce a graph of a line (linear)
SC2: I can give an example of a pattern that will produce a graph of a curve (non-linear).
SC3: I can make connections between the numbers in my table of values and the points on the graph.

LG2: We are learning to generate data and use graphical representations to determine a pattern in the data.

SC1: I can plot points to generate a graph by hand.
SC2: I can enter data into a spreadsheet or graphing program to generate a graph.
SC3: I can use technology to show lines or curves that represent the pattern I am exploring.

LG3: We are learning to use graphs to explore density and infinity.

SC1: I can use technology to identify points that fall between 2 whole numbers on my graph.
SC2: I can explain how there are infinite possibilities for reaching a target number.

CONSIDERATIONS THROUGHOUT THE LESSON

Differentiated Instruction and Universal Design for Learning

Use Flexible Groupings (eg: Visibly Random Groupings, heterogeneous groupings) to create inclusive small groups.
Allow for student choice in picking a target number to work with.
Allow for student choice in picking an application question with which to link their learning.
Display a multiplication chart in the room to assist students with identifying factors of numbers.
Previewing vocabulary for English Language Learners and students with special education needs would be beneficial. (Examples of terms that may be included: x-axis, y-axis, line, curve, sum, product, x-intercept, y-intercept)

Assessment

Throughout the lesson, the teacher will be listening for students to be correctly and effectively using mathematical language to describe their mathematical thinking.
Listen for understanding of differences between linear and non-linear patterns and graphs, and the concept of infinite possibilities. (See Appendix for observation sheet.)
Check point - Generating linear and non-linear graphs (see Action section)
Are students seeing the connections between the combinations of numbers that they generated and the points on the graphs?
Exit ticket (see Consolidation section)

RESOURCES AND LEARNING ENVIRONMENT

Educator Resources Needed

Multiplication chart to display
Copy of Minds-On graphs to display
Access to graphing technology (eg: Excel, Google Sheets, Desmos, Graphing Calculator)

Student Materials Needed

Each group needs 2 page protectors with a copy of the graphing template inside (see Appendix)
Markers
Graphing technology (at least 1 device per group of 3)

Learning Environment Considerations

During this lesson, students will be asked to focus on building relationship skills and developing critical and creative mathematical thinking.
From the Curriculum Context, building healthy relationships and developing mathematical thinking will include:

- Listening attentively
- Considering other ideas and perspectives
- Using cooperation and collaboration skills
- Applying strategies such as:
    - seeking opportunities to help others
    - working as part of a team and playing different roles (e.g., leader, scribe or illustrator, data collector, observer) that contribute to outcomes in different ways
- Making connections
- Communicating effectively
- Applying strategies such as:
    - using charts, diagrams, and representations to help identify connections and interrelationships

The Minds-On section starts with students working with someone who sits near them. Teacher may need to allow for groups of three depending upon class set-up or numbers.

In the Action section, students will work in groups of three, generated in a visibly random or heterogeneous way, to complete the exploration and graphing activities.

For part of the Consolidation section, students will return to their groups from the Action section.

Previewing vocabulary for students who are English Language Learners and students with special education needs would be beneficial. (Examples of terms that may be included:

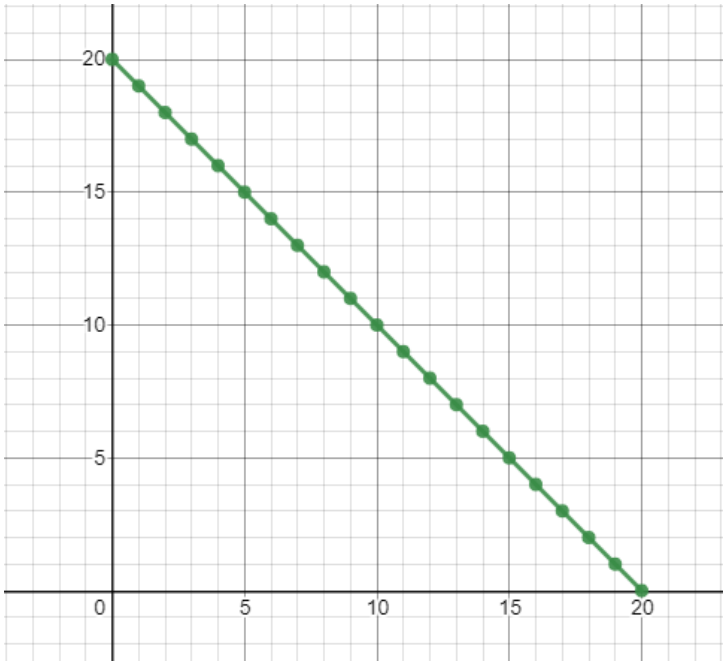
x-axis, y-axis, line, curve, sum, product). The use of a word wall with visuals would be one way to ensure this is in place and to provide continued support for all learners.

Consider “de-fronting” the classroom, allowing students to sit in groups rather than in rows facing the front of the room. This helps send the message to students that the space is safe and it is not necessary to be perfect.

LESSON CONTENT

Minds-On (10 Minutes)

Start this lesson by displaying these graphs for students to view. Ask the students “What do you notice? What do you wonder?”



<This graph shows a linear graph in Quadrant I with x-intercept (20,0) and y-intercept (0,20). Points are marked along the line for every whole number value of x.>

<This graph shows a non-linear graph in Quadrant I. Points are marked along the curve at (1,20), (2,10), (4,5), (5,4), (10,2), (20,1).>

Collect and display student answers.
If the following don't emerge as part of your "What do you notice? What do you wonder?", ask a follow up question of "What do you think these points could represent? How do you think they were generated?" Give students an opportunity to Think-Pair-Share with someone sitting near them.

Answer:
In graph 1, the points are generated by finding the number pairs that add to 20 (eg: 1+19 = 20, so the point (1,19) is on the graph).
In graph 2, the points are generated by finding the number pairs that multiply to 20 (eg: (2)(10) = 20 so the point (2,10) is on the graph.

Ask students what they notice as being the same and being different between these graphs. Again, give students an opportunity to Think-Pair-Share with someone sitting near them.

Sample answers:

- both graphs have a scale of 0 to 20 on both the x- and y-axis
- both graphs are going down to the right
- as the first number gets bigger the second number gets smaller
- the first graph has more points than the second graph

- the first graph is straight and the second graph is curved
- the straight line has an x-intercept and a y-intercept but the curve has neither

Ask students if they think that adding 2 numbers to get a certain number and multiplying 2 numbers to get a certain number will always produce graphs that look like this.

Share learning goals with students.

Action (50-60 Minutes)

Place students in visibly random groups of 3-4 students. Have each group pick an even number between 20 and 100 to investigate.

Each group needs to generate 2 tables of values with “1st number” in the x column and “2nd number” in the y column. In the first table, the group should come up with a list of number pairs that when added together result in the number they chose to investigate. In the second table, the group should come up with a list of number pairs that when multiplied together result in the number they chose to investigate. If groups are struggling to come up with number pairs, the teacher may want to suggest working through numbers in a logical manner, such as start with 1, then try 2, then try 3, and so on. Students should also be encouraged to include both possibilities for factors (ie: 4 x 5 and 5 x 4 should both be written in their table and plotted on the graph). It is likely that students will stick with whole numbers during this part of the activity. If some groups do include fractions or decimals, you can use this as a jumping off point for the second activity involving technology (see below for details).

Students then need to graph their results on the templates provided by the teacher. These templates need to be placed into a page protector, and students should be drawing their graphs right on the page protector. As the students finish their graphs, they should hand their page protectors in to the teacher.

Checkpoint: The teacher should check the graphs for accuracy. If problems are identified, feedback should be given and the group should make revisions and hand the graph back in.

Once graphs are collected, the teacher should gather students for a whole class discussion. Ask the students what they noticed when creating their graphs.

Sample answer:

- The graph of number pairs that add to our number was straight and the graph of number pairs that multiply was curved, just like it was with 20.

- It is easier to determine pairs of numbers that add to a number than pairs that multiply to that number.
- There are more points plotted to make the straight line than the curved line.

The teacher should now remove the paper from inside the page protectors and stack all of the page protectors with the linear graphs on top of each other and show the pile to students. They should be able to see that all of the graphs look the same, they have just been shifted. Have a discussion about what the students notice. Vocabulary that could be included in the discussion could be steepness, x-intercept, y-intercept, translated, parallel. Highlight for students that these graphs can all be described by the equation $x + y = k$. You may want to introduce the idea that to get from one graph to the next you could do a vertical translation or a horizontal translation. This will help to set the stage for translation work in grade 10.

Repeat this procedure with the non-linear graphs. Again, ask the students to describe what they notice. As part of the discussion the teacher could draw attention to the distance between the origin and where the graph turns to the right. Ask students what they notice about this point. They might notice that this occurs when the factors start to repeat, but in the opposite order, so it identifies where the factor pairs are closest together. Highlight for the students that these graphs can all be described by the equation $xy = k$.

Ask students why they think that when we added numbers together we got a linear graph and when we multiplied numbers together we got a curve. Lead the discussion towards the observation that when we are finding numbers to add together the numbers change in the same way (e.g.: in moving from our combination of 1 + 19 to 2 + 18, the first number increased by 1 while the second number decreased by 1). When we find numbers to multiply together the numbers change in different ways (eg: in moving from 1 x 20 to 2 x 10 we increased the first number by 1 and decreased the second number by 10. When we move from 2 x 10 to 4 x 5, we increase the first number by 2 and decrease the second number by 5). Link this to the patterns one can see in the differences between linear and non-linear relations.

Ask students whether or not they think they have found all of the possible combinations for their numbers. Lead the discussion towards the realization that we could also use fractions or decimal numbers to reach our desired number, it just might be harder to come up with the combinations off the top of our heads. To help come up with possible combinations, we are going to use technology.

Extension Opportunity (before introducing graphing technology)

- Teachers could ask students to predict what they think the graph will look like if we extend the axes to include negative numbers. How will it affect the graph? After they

use technology to graph the relations, ask the students to reflect upon what they see. Were they surprised? Or did the technology show them what they expected?

Have students return to their groups from the first part of the Action section. They need to enter the tables of values they created into a graphing program (eg: graphing calculator, Excel, Google Sheets, Desmos, etc.). They need to create a graph for each of their tables, and then generate the line or curve of best fit. Depending upon the program used, the students may need to perform a regression to generate the equation (eg: graphing calculator) or they may be able to enter the equation in the $x + y = k$ and $xy = k$ forms (eg: Desmos). Once they have generated the graph, they need to pick 2 consecutive points from their table of values and locate them on their graph. We will call these a and b. Using a trace feature, they need to identify the coordinates of 2 points that fall between a and b. Reassure the students that it is fine to have decimal or fractional values. Once they have identified the 2 new points (we will call these c and d) and added them to their table of values, have them find the coordinates of 2 points between c and d. They can call these e and f, and can add these to their table of values. Now have them find the coordinates of 2 points between e and f. They can call these g and h and add them to their table.

Questions for reflection:

- Is there a set amount of number pairs that add to or multiply to each number?
  Answer: There are infinitely many combinations available because of the density of the real numbers.*Density is the concept that between any given two real numbers, there will always be another real number. Thus, there are infinitely many real numbers between any two real numbers. (*from the Curriculum glossary)

- What do you notice about the coordinates of the points c, d, e, f, g, and h?
  Answer: As we look for new points between other points that are close together, we need to include more decimal places to describe the coordinates.

- If you keep picking 2 points that are closer and closer together, will you always be able to find 2 new points between them?
  Answer: Yes, because real numbers are dense, you will always be able to find another number between the two that you already have.

- Why do all the linear graphs have the x- and y-intercepts they do? How do these points relate to their equation?
  Answer: The x- and y- intercepts correspond to the situations where we add 0 to either x or y to get the k value. This means that the x-intercept will always be (k,0) and the y-intercept will always be (0,k).

- Do the non-linear graphs have x- and y-intercepts? Why do you think that?

Answer: The graphs of xy = k do not have x- or y-intercepts because we cannot multiply x or y by 0 to get the value of k. If x or y is 0, there is no solution to the equation (eg: 0 = k).

- What do you notice happens to each of the non-linear graphs near the x- and y-axes?
  Answer: The curve gets really close to the y-axis when x is really close to 0 and it gets really close to the x-axis when y is really close to 0, but x and y can never actually be 0. This is an example of a limit.

After students complete their 2 original graphs, have them explore how the graphs and their number combinations change if they work with 2x + 2y = k rather than x + y = k.

Question for reflection:
-If you have a list of combinations for x + y = k, what could you do to them to get combinations that would work for 2x + 2y = k?
Answer: if you divided each x and y value by 2, you would get your new combinations. Follow this up by asking students why this makes sense.

Extension Opportunities

- Teachers could ask students to explore combinations of numbers that add together to give you 1, using fractions or decimals between 0 and 1.
- Teachers could ask students to work with x - y = k. How does this affect the graph?

Consolidation (20 Minutes)

The teacher should bring the whole class together to debrief what has been discovered. Points to include could be:

- there are infinite combinations of numbers that can be used to add or multiply to a given number, if we expand beyond using whole numbers
- we can find those combinations using the graphs x + y = k and xy = k
- the graph x + y = k is linear while the graph of xy = k is non-linear

Students should now return to their groups from the Action section to work on one of the following problems.

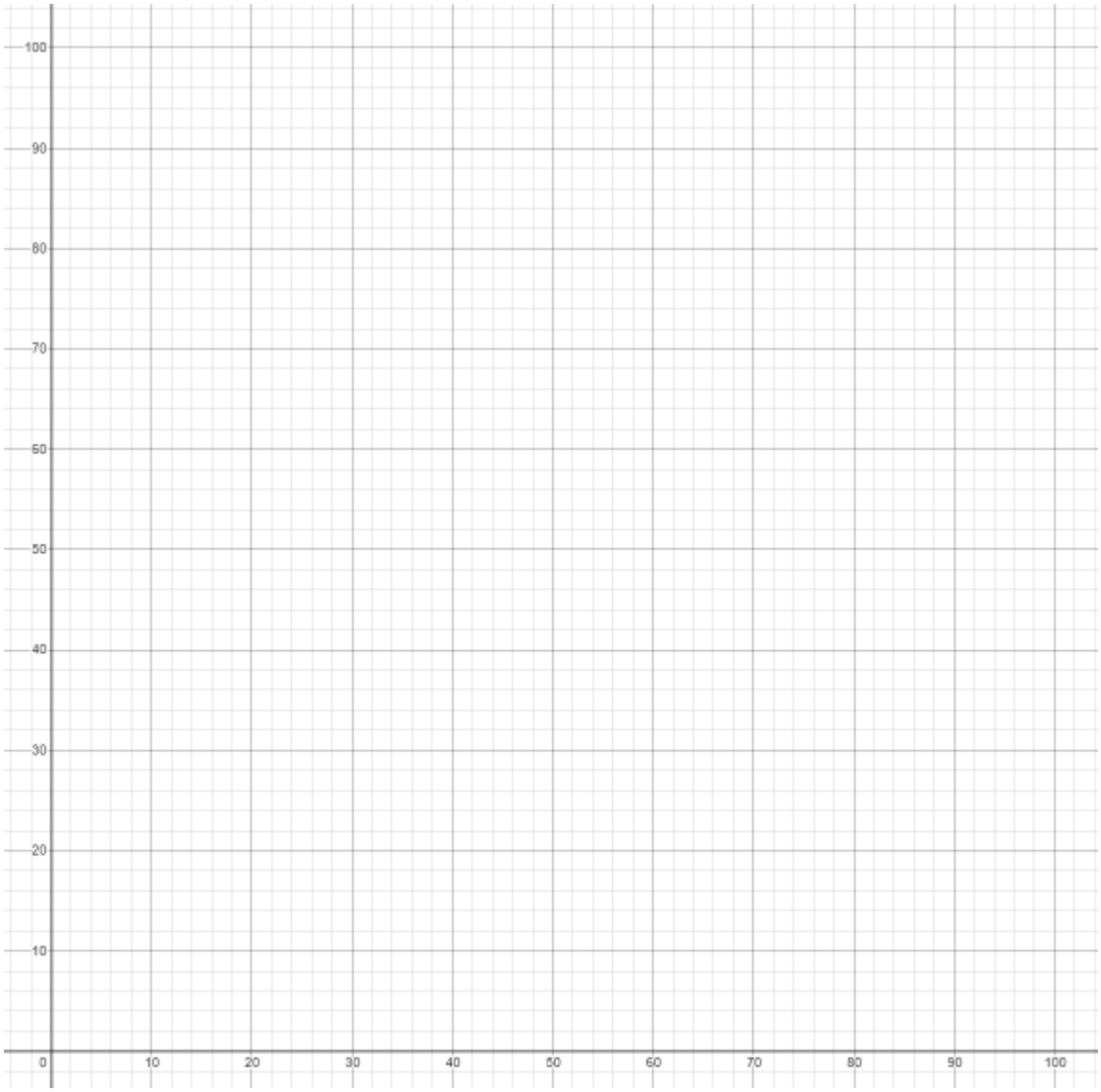
a) You have $15 to spend at a restaurant. You want to buy an entree and a dessert. How much money could you spend on each item? Suggest some items that you might want to buy.
b) You want to build a small vegetable garden that has an area of 10 m^2. What could the dimensions be? Will you be limited if you decide to edge your garden with 0.25 m long bricks?
c) You want to make a rectangular space on the wall of your room to display photos. You have 18 feet of LED lights to use as a border. What could the dimensions of your rectangle be? Which option would you choose?

After the group work on the problems, each student should complete the Exit Ticket (copy included in the Appendix):

a) Write one sentence to sum up what you learned in math class today.
b) Write one question that could be answered using what you learned in math class today.

Appendix:

Template for Action Section



<This is a coordinate grid that spans from 0 to 100 on both the x and y axis. The scale counts by 10's, but the lines are shown for counting by 2's.>

Template for Conversation Observation Sheet

Student Name	Use of mathematical language in discussions	Can describe difference between linear and non-linear relations	Can describe concept of infinite possibilities for points
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
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	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4
	NA 1 2 3 4	NA 1 2 3 4	NA 1 2 3 4

NA - Not observed
1 - Level 1
2 - Level 2
3 - Level 3
4 - Level 4

Exit Card:

Name: _____

Write one sentence to sum up what you learned in math class today.
Write one question that could be answered using what you learned in math class today.