## LESSON DETAILS

### Whose Triangle Theorem is it Anyway?

### Lesson Summary

This lesson gives students an opportunity to explore properties of right triangles and similar triangles, and also explore and ask questions about the long history of the side length relationship of right triangles (also known as the Pythagorean Theorem).

### Grade: 9

### Big Ideas

Triangle properties, research of development of math concepts

### Learning Expectations

**AA1.** develop and explore a variety of social-emotional learning skills in a context that supports and reflects their learning in connection with the expectations across all other strands

- Developing critical and creative mathematical thinking
- Developing a healthy mathematical identity through building self-awareness

**A1.** apply the mathematical processes to develop a conceptual understanding of, and procedural fluency with, the mathematics they are learning

- Reasoning and proving
- Reflecting

**A2.** make connections between mathematics and various knowledge systems, their lived experiences, and various real-life applications of mathematics, including careers

**B3.** apply an understanding of rational numbers, ratios, rates, percentages, and proportions, in various mathematical contexts, and to solve problems

> **B3.5** pose and solve problems involving rates, percentages, and proportions in various contexts, including contexts connected to real-life applications of data, measurement, geometry, linear relations, and financial literacy

**C1.** demonstrate an understanding of the development and use of algebraic concepts and of their connection to numbers, using various tools and representations

**C1.4** simplify algebraic expressions by applying properties of operations of numbers, using various representations and tools, in different contexts

**E1.** demonstrate an understanding of the development and use of geometric and measurement relationships, and apply these relationships to solve problems, including problems involving real-life situations

**E1.1** research a geometric concept or a measurement system to tell a story about its development and use in a specific culture or community, and describe its relevance in connection to careers and to other disciplines

**E1.4** show how changing one or more dimensions of a two-dimensional shape and a three-dimensional object affects perimeter/circumference, area, surface area, and volume, using technology when appropriate

**E1.5** solve problems involving the side–length relationship for right triangles in real-life situations, including problems that involve composite shapes

## Cross Curricular Connections

This lesson builds on the historical inquiry process used in Grades 7 and 8 History, and prepares students to use this process in Grade 10 History.

## Learning Goals and Success Criteria:

These are some suggested learning goals for this lesson. Ideally, these should be co-created with students when beginning the different parts of the investigation.

LG1: We are learning to develop and use geometric relationships.

SC1: I can calculate the area and perimeter of triangles.

SC2: I can solve for the unknown side length of a right triangle (given some combination of perimeter, side lengths, and/or area).

SC3: I can prove how changing one dimension affects the area and perimeter of a right triangle.

LG2: We are learning to develop our reasoning and ability to prove mathematical relationships.

SC1: I can make a conjecture about a rule for a geometric relationship and test it to determine its validity.

SC2: I can determine whether our rule works for only a specific case or works in general.

LG3: We are learning to research the side-length relationship for right triangles and tell a story about its development and its use in different cultures.

SC1: I can ask and answer critical questions about research we have read.

SC2: I can reflect and make connections between math and society.
SC3: I can be critical of the sources of our research.

## CONSIDERATIONS THROUGHOUT THE LESSON

### Differentiated Instruction and Universal Design for Learning

- **Previewing vocabulary** - The class will develop definitions at the beginning of the lesson (see Minds-On section).
- **Pacing** - Each small group will work at their own pace, and can be given support and/or extensions, or the next task, as needed.
- **Use of manipulatives** virtual -- geometry software: The teacher could print out a paper version of the Gougu Theorem -- or get students to cut out and create their own.
- **Choice in topic** for further research.
- **Chunking of tasks** - The exploration of triangle properties can be broken into two parts (side length and perimeter, side length and area).
- **Multiple ways** for students to communicate their findings -- Students could use a voice note, interview with the teacher, or write them down.
- **Assistive technology** - Students can use voice-to-text, text-to-voice, and organizers in apps such as Google Read&Write to help with their research and synthesis.

### Assessment

Educators can build in **assessment for learning** (observations and conversations) during the vocabulary preview, the “Finding Patterns using the Triangle Chart” activity, and the Research Stations task. In all of these activities, students are working in small, visibly random groups on vertical surfaces, so the teacher will be able to check in with all of the groups as they work together.

The teacher will also be able to have **checkpoints** for the students as they rotate through the Research Stations task. They can explain and show their answers to the teacher before being given the next task, to ensure understanding.

At the later part of the lesson, students will complete some research of their own and reflect on what they have learned. This is a good opportunity for **assessment of learning** (see Consolidation section).

## RESOURCES AND LEARNING ENVIRONMENT

### Educator Resources Needed

- Document MTH1W History of the Side Length Relationship in Right Triangles
- Grid paper to make manipulatives (for Gougu Theorem)
- Pieces of string (for Egyptians and Triangles)

### Student Materials Needed

- Computer, tablet or phone to run the geometry software (or alternatively, students can create cutout shapes from grid paper to measure)
- Computer, tablet or phone to do further research on side relationship in right triangles

### Learning Environment Considerations

**Small Groups** - When working on group activities, students should be working in visibly random groups of 2-4 (with an ideal group size of 3). More than 3 people working around a computer makes it challenging for all members of the group to take part (although finding dimensions of similar triangles is a small part of the lesson). Randomizing the groups, and being transparent about the randomization, contributes to a collaborative environment. Students will eventually work with most of their classmates, and as a result will be more comfortable in the class and be more willing to take mathematical risks. If the lesson takes more than one class period, you may want to re-randomize the groups for the next period, assuming students aren't midway through a task.

**Non-permanent vertical surfaces (NPVS)** - When students are working in their small groups, ideally they should be working at NPVS. The non-permanency of the surface (ie. a whiteboard or blackboard vs. chart paper or notebooks) means that students are more willing to write things down - they can be free to make mistakes or change their strategy. The fact that the surfaces the students are working on are vertical means that it is easy for the teacher to see their work at a glance, and move students along or give a small "just-in-time lesson" if needed. It also allows for knowledge mobility within the whole group. Students are able to see what another group is doing and go ask them a question, or they might consider how they might make use of ideas they have seen to complement their own ideas.

**Desk arrangement** - Desks should be arranged in small groups in the classroom. As the majority of their work (except for the final research and consolidation) will be working in groups at NPVS around the room, the room can be "de-fronted". There is no need for all students' desks to be facing one side of the classroom. This lends to the collaborative environment in the classroom.

**Whole class consolidation** - Be aware of students' lived experiences when consolidating the lesson, especially when they reflect on how we have developed and given credit for math theorems throughout history. Be prepared for questions or comments about why history has favoured a euro-centric perspective, and address comments in a meaningful way through discussion.

## LESSON CONTENT

**Minds-On** (20-30 Minutes)

- Describe the educator steps taken to introduce the lesson, activate student thinking, and connect to prior learning
- Include actions the students will take during this section
- Outline learning goal(s) for students

**Vocabulary Review**

At the beginning of the class, the group should review the terms: leg, hypotenuse, similar triangle, perimeter (of a triangle), area (of a triangle) and ratio.

Split the class into 6 groups (one for each term). In 6 spots throughout the room, preferably on non-permanent vertical surfaces (VNPS), write the six vocabulary terms down. If splitting into 6 makes your groups too large, feel free to adjust and double up on some of the definitions, creating two spots for some of the terms. Then have each group start with one term, and give 5 minutes for them to create a definition for the term (or until each group has an initial definition). Then, have each group cycle through all the rest of the terms. This will be quicker, around a minute or two per term. They can add to the definition, create a diagram, and example or add a star to an idea they like. Then the group cycles back to their initial term, and chooses their preferred definition/diagram to share back with the group. During this vocabulary consolidation, the teacher can ask questions of students to ensure the final version of the definition is correct. The class's preferred definitions should then be left visible somewhere in the room for the students to access for the rest of the lesson.

**Filling out the Triangle Chart**

Once the class understands the definitions of what they will be measuring, they can collect the following data using a table such as the one shown. They can work in visibly random small groups (2-3) to use geometry software to determine 4 similar right triangles. Give students the instructions that one of their similar triangles should have side lengths twice as long as another one.

Here are two options for pre-made right triangles created using geometry software that students can use:
https://www.geogebra.org/calculator/vzrffj52
https://www.desmos.com/geometry/6gvtco58kl

| | Leg 1 | $(\text{Leg } 1)^2$ | Leg 2 | $(\text{Leg } 2)^2$ | Hypotenuse | $(\text{Hypotenuse})^2$ | Triangle | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | Perimeter | Area |
| Triangle 1 | | | | | | | | |
| Triangle 2 | | | | | | | | |
| Triangle 3 | | | | | | | | |
| Triangle 4 | | | | | | | | |

**Action** (70-85 Minutes)

**Co-Creating Success Criteria**
At the beginning of this section, have students brainstorm success criteria for determining geometric relationships. Ask students (in visibly random small groups of 2-4) to answer the questions: "What matters and what is most important when determining a geometric relationship?" and "What matters and what is most important when determining if your rule works for all cases, or only special cases?". Groups can each share which of their criteria they feel is most important, and as a class you can generate a short list for each. The teacher can help guide the conversation by asking questions about patterns and proof such as, "How can we tell if it works for all cases?", and by collecting a list (which is not too long) of success criteria for each section.

**Finding Patterns in the Triangle Chart**
The next task is for students to work in the same small groups to develop some geometric relationships for their triangles. You may wish to ask them one question at a time, and even one part of each question at a time.

1. Develop a relationship between $(\text{Leg } 1)^2$, $(\text{Leg } 2)^2$ and $(\text{Hypotenuse})^2$. Does this work for all triangles, or just your set of triangles? How do you know? (This is meant as a jumping off point for creating and proving conjectures, since students would have

explored the Pythagorean theorem in grade 8. It also serves as a jumping off point for the history part of the lesson later. Once students have come up with the relationship between the squares of the side lengths, they can spend more time developing their own relationships in 2.)

2. Develop a relationship between side length or perimeter and the area of your triangles. Does this work for all triangles, or just your set of triangles? How do you know?
   Ex 1. Investigate what happens to the area and perimeter of one of your similar triangles when you double each of its side lengths.
   Ex 2. Investigate what happens to the area and perimeter of one of your similar triangles when you increase/decrease each of its side lengths by the same factor.

If students are having trouble finding patterns with their triangle table, encourage them to create other similar triangles using the software with sides doubled, and/or tripled. That will allow them to see ratios more clearly.
Examples of conjectures they might end up coming up with are:

a) If I double the side length of a similar right triangle, its area is multiplied by four.
b) If I double or triple etc. the side lengths of my triangle, its perimeter also doubles or triples.
c) If I double the perimeter, the area is multiplied by four.

Once a group has created a conjecture about a relationship, they should be encouraged to test it using another group's triangles. This will help them determine if their conjecture works only for their set of triangles, or if it may work for all right triangles.

**Mini-Consolidation - Finding Patterns**
Before moving on to the research stations, have students consolidate their review of vocabulary and work on their conjectures by writing a note for themselves about what they learned. This can be done by creating examples of their rules and annotating them so they will be able to understand what they were doing later.

**Research Stations**
Once all students have developed the side-length relationship for right triangles (you can determine how far you want them to go with the perimeter and area relationship based on time), then introduce the idea about how this relationship has been known for thousands of years, and has been proved hundreds of different ways, by letting students rotate through several research stations.

There are six stations, each relating to one era in the history of the Pythagorean theorem, in this document [MTH1W History of the Side Length Relationship in Right Triangles]

In each station students should read the information in the paragraph, and in their small group, answer the questions given. Some questions in each station are related to the mathematics and some related to the history. They can work out their answers on a group document (digital, whiteboards, paper) and/or give the answers verbally as well. Once a group has finished a research station, they can move on to the next, taking their notes with them. There is no order necessarily (they are in chronological order in the resource).

The teacher may decide if a follow up whole group sharing is needed or valuable.

**Extension Opportunities**

- Students can graph the relationship they have discovered between side length or perimeter and area of different sized similar triangles
- Students can find another set of properties and try to determine a relationship in their triangles
- Students can find another proof of the Pythagorean theorem (there are many easily found on the internet) to analyse and understand

**Consolidation** (30-45 Minutes)

**Co-Creating Success Criteria**
At the beginning of this section, have students brainstorm success criteria for answering the research questions. Ask students to answer (in visibly random small groups of 2-4) the questions: “What matters and what is most important when evaluating sources?”, “What matters and what is most important when finding interesting information?” and “What matters and what is most important when making connections between math and society?” Groups can each share which of their criteria they feel is most important, and as a class you can generate a short list for each. The teacher can help guide the conversation by asking questions about internet research such as, “What makes a source credible?”, and by collecting a list of success criteria which is not too long for each section.

**Going Deeper into the History**
After working through all of the stations, each student will use the internet to further research the information from one of the paragraphs, and generate one new finding and one more question relating to the artifact or mathematician. They will also briefly evaluate the source they found, and give an example of why that culture used right triangles. Make sure to get students to include the URL of the website from which they found their information.
Considerations:

- Include relevant and detailed information that pertains to the consolidation of the lesson of the lesson
- Revisit learning goals and refine success criteria
- Include opportunities for student reflection as well as any end of lesson assessment check ins for educators (e.g., exit ticket, quick poll, etc.)

**Suggested questions for research and reflection:**
Use the internet to read some more about one of the Research Station topics of your choice.

1. Find out one reason about why the people found triangles important in their culture (why did they have the need to find the hypotenuse of a triangle?)
2. Find another interesting piece of information about your topic.
3. Write down a new question you have about the topic as a result of your research.
4. Evaluate the website from which you found your information. Does it look reputable? How do you know? Write down the URL of the website where you found your information.

**Suggested rubric for research and reflection:**

| Areas of Strength | | Areas for Growth |
|---|---|---|
| | I can ask a thoughtful question about the research I have read about right triangles throughout history. | |
| | I can reflect and make connections between math and society. I understand some different ways right triangles were important in many ancient societies. | |
| | I can be critical of the sources of my research. I can identify if a website seems to be a good source. | |

**Suggested Exit ticket with reflection from research:**

You have just researched one of the more memorable theorems in mathematics.

1. What have you learned about how different cultures and eras in math used this theorem?
2. What have you learned about how theorems are named in mathematics? Is it fair? Does it matter who gets the credit?
3. Using what you now know about right triangles and their uses throughout history, create (and solve) a “word problem” related to some of the history you have learned.
4. Why might the “3-4-5 configuration” be so prominent in historical documentation of right triangles? Why this rather than, say, “5-12-13” or some other right triangle?

**MTH1W History of the Side Length Relationship in Right Triangles**

**Egyptians and Right Triangles**
Location: Ancient Egypt Time: 3000 BC
https://www.radford.edu/~wacase/Math%20135%20Pythagorean%20Theorem.pdf

Many sources say ancient Egyptians used circular ropes with 12 evenly spaced knots on them in order to create right angles for building. However, there is no tangible proof that was the case -- and in many cases, scholars argue there are easier ways to create a right angle. Use the string provided to create your own version of this rope, use the rope to create a right triangle, and then use it to try to create a square figure (like an outline of a miniature building).

**Q1: What other ways are there to use string (or strings) to create a right angle?**
**Q2: Why do you think there are so many sources that state Egyptians used a rope to make right angles, even though there's no evidence that it is true? What other myths or urban legends can you think of which are similar?**

**Plimpton 322**
Location: Ancient Babylonia (Modern Iraq) Time: 1800 BC
https://blogs.scientificamerican.com/roots-of-unity/dont-fall-for-babylonian-trigonometry-hype
https://www.ams.org/publicoutreach/happ5-history.pdf
https://www.maa.org/sites/default/files/pdf/upload_library/22/Ford/Robson105-120.pdf

Plimpton 322 is the name given to a clay tablet found near what was the Ancient city of Larsa (now Tell Senkereh, Iraq). George Plimpton bought the tablet in 1923. In the 1940's, a Brown University historian Otto Neugebauer and his assistant Abraham Sachs figured out that the entries in the tablet were Pythagorean triples (whole numbers, which satisfy the equation $c^2 = a^2 + b^2$). There are many Pythagorean triples in the tablet, however some lines have mistakes in them. That, and how the tablet is organized in a table makes historians think that the tablet might have been a study exercise for a student.

**Q1: What are some Pythagorean triples that you know? Find 2 or 3 without using the internet.**

**Q2: How does it make you feel that people might have been doing math worksheets back in 1800 BC?**

**Baudhayana and Katyayana Sulbasutras**

Location: Ancient India     Time: 800-200 BC

https://mathshistory.st-andrews.ac.uk/Projects/Pearce/chapter-5/

https://www.cuemath.com/learn/baudhayana/

https://mathshistory.st-andrews.ac.uk/HistTopics/Indian_sulbasutras/

There are two texts from Ancient India which reference what is now referred to as the Pythagorean theorem. One, the Baudhayana Sulbasutra, created around 800 BC, references a special case of the theorem, for finding the diagonal of a square, not a rectangle. It is stated this way: "The rope which is stretched across the diagonal of a square produces an area double the size of the original square." The second, the Katyayana Sulbasutra, created around 200 BC, states a more general case of the theorem: "The rope which is stretched along the length of the diagonal of a rectangle produces an area which the vertical and horizontal sides make together." The Sulbasutras were appendices to the Vedas, religious books. The people practicing the Vedic religion would perform rituals at altars. In order for the rituals to be successful, the altars needed to be precisely made.

**Q1: How can you find the rule for calculating the length of the diagonal of any square? How could you prove it to be true for all squares?**
**Q2: Why do you think people would have figured out how to get the diagonal of a square before a rectangle?**
**Q3: What do you think is more valuable -- finding out a relationship is true, or proving it to be true for all cases? Why? (There is no proof in the Sulbasutra, just a statement.)**

**Pythagoras**

Location: Southern Italy     Time: 570-490 BC

https://plato.stanford.edu/entries/pythagoras/#WasPytMatCos

The name “Pythagorean Theorem” has been used for centuries to describe the side length relationship in right triangles. However, there is no evidence that Pythagoras wrote anything, and there are no accounts from people at the time about his mathematics. The only real evidence that associates Pythagoras with the theorem that bears his name is a quote from Proclus almost a thousand years later, who was commenting on Euclid's proof of the theorem. The quote reads, "If we listen to those who wish to investigate ancient history, it is possible to find them referring this theorem back to Pythagoras and saying that he sacrificed an ox upon its discovery". Where Proclus got this information is uncertain, although it could be from two lines of verse (author possibly from around 400 BC). Proclus also talks about Pythagoras "knowing the truth of the theorem", but with no mention of him proving it, which is quite possible, as several societies had known about it hundreds of years before him.

**Q1: How do you think it is possible that there is no actual proof of Pythagoras proving the Pythagorean theorem?**
**Q2: What matters to you in how math theorems are named? What criteria do you think are important?**

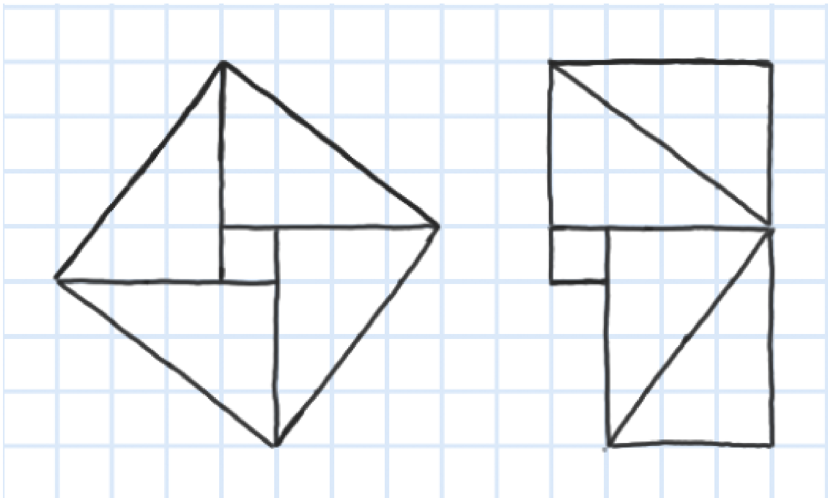
**Euclid**
Location: Alexandria (Present day Egypt)    Time: 300 BC
https://mathcs.clarku.edu/~djoyce/elements/bookI/propI47.html
https://americanhistory.si.edu/collections/search/object/nmah_694620
https://link.springer.com/article/10.1057/jt.2009.16

Euclid did a lot of work writing about math, and his works are some of the oldest that still survive today. His "Elements" is a text composed of 13 books, which start with simple definitions - the very first is "a point is that which has no part". After the definitions there are postulates, common notions, and then propositions, where he proves concepts to be true, only using facts which he has proven earlier in his text. He proves the Pythagorean theorem twice (two completely different ways) in his work.
https://gallica.bnf.fr/ark:/12148/bpt6k1521328b/f51.item French link to the Elements
https://mathcs.clarku.edu/~djoyce/elements/toc.html English link to the Elements

**Q1: Use the internet to check out the first book of Euclid's elements. Try his first proposition creating an equilateral triangle. What do you find interesting? What did you find challenging?**
**Q2: After looking at Euclid's Elements a little, why do you think it is so valuable as a historical math text? What other old documents like it can you think of that are important in other subjects other than math?**

**The Gougu Rule**
Location: Ancient China Time: 221-206 BC
https://www.ck12.org/book/geometry-pythagorean-theorem/section/3.1/
https://www.maa.org/book/export/html/59135
https://avserzhen.files.wordpress.com/2016/06/astronomy-and-mathematic-in-ancient-china.pdf

The Gougu Rule was developed in China independently of mathematicians in the Mediterranean area. The word Gou means hook, referencing the shorter leg of the triangle, and the word gu means thigh, referencing the longer leg of the triangle. The name given to the hypotenuse is “xian”, which means bowstring.



Here is a simplified diagram of a visual proof of the Gougu rule. This image was added several hundred years later to the text, probably by someone trying to illustrate the rule that had been stated, possibly trying to recreate a diagram that was lost to time. According to scholars, the person who was adding these images may not have understood them -- there are some images which attempt to illustrate the rule but don’t make sense.

**Q1: How can you figure out how the diagram shows the side length relationship in right triangles? Does this diagram work for any side length?**
**Q2: Why do you think the text containing the Gougu Rule has been added-to and altered over the years?**